

On Approximate Controllability of Impulsive Linear Evolution Equations

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In this paper, we study an approximate controllability for the impulsive linear evolution equations in Hilbert spaces. The necessary and sufficient conditions for approximate controllability in terms of resolvent operators are given. An example is provided to illustrate the application of the obtained results.

Keywords: Approximate controllability, impulsive equation, evolution equation, resolvent condition

1. Introduction

In recent years, there has been growing interests towards the study of controllability of impulsive systems, that is, the systems in which the system-state is subject to impulse at discrete time points. This topic has popularity and quite broad literature, see for example, [2], [3], [4]-[11]. The present paper studies the necessary and sufficient conditions for approximate controllability of dynamical systems described by linear impulsive differential equations in Hilbert spaces, under the basic assumption that the operator A acting on the state is the infinitesimal generator of a strongly continuous semigroup. Approximate controllability means we can control a transfer from an arbitrary point to a small neighborhood of any other point. It can be explained by the fact that in infinite-dimensional spaces, there are linear non closed subspaces, see [12]. By looking at the approximate controllability problem as the limit of optimal control problems and reformulating the optimal control problem in terms of the convergence of resolvent operators, we found the necessary and sufficient conditions for the for approximate controllability of the impulsive linear evolution systems. The condition in terms of resolvent is easy to use and it is used in the number of articles devoted to the approximate controllability for different semilinear differential equations. In the absence of impulses, so-called resolvent condition is equivalent to the approximate controllability of the associated linear part of the semilinear evolution control system (see [14], [15]), but the problem becomes complicated in the presence of impulses.

In this article, we investigate approximate controllability of the following linear evolution systems with impulse effects

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in [0, b] \setminus \{t_1, \dots, t_p\}, \\ \Delta x(t_k) = C_k x(t_k) + D_k v_k, & t = t_k, \quad k = 1, \dots, p, \\ x(0) = x_0, \end{cases} \quad (1)$$

where the state variable $x(\cdot)$ takes values in Hilbert space H with the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. The control function $u(\cdot)$ is given in $L^2([0, b], U)$, a Hilbert space of admissible control functions with U as a Hilbert space, $v_k \in U$, $k = 1, \dots, p$. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ in H , $B \in L(U, H)$, $C_k \in L(H, H)$, $D_k \in L(U, H)$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^\pm) = \lim_{h \rightarrow 0^\pm} x(t_k + h)$ with discontinuity points t_k , $k = 1, \dots, p$, $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = b$. It is assumed that $x(t_k^-) = x(t_k)$.

Lemma 1 *The mild solution of (1) is given by*

$$\begin{aligned} x(t) &= S(t)x(0) + \int_0^t S(t-s)Bu(s)ds, \quad 0 \leq t \leq t_1, \\ x(t) &= S(t-t_k)x(t_k^+) + \int_{t_k}^t S(t-s)Bu(s)ds, \quad t_k < t \leq t_{k+1}, \quad k = 1, 2, \dots, p, \end{aligned}$$

where

$$\begin{aligned} x(t_k^+) &= \prod_{j=k}^1 S_C(t_j, t_{j-1})x_0 + \sum_{i=1}^k \prod_{j=k}^{i+1} S_C(t_j, t_{j-1}) \int_{t_{i-1}}^{t_i} S_C(t_i, s)Bu(s)ds \\ &\quad + \sum_{i=2}^k \prod_{j=k}^i S_C(t_j, t_{j-1})D_{i-1}u_{i-1} + D_k u_k, \quad S_C(t_j, t_{j-1}) := (I + C_j)S(t_j - t_{j-1}). \end{aligned}$$

Proof. The finite dimensional case is proved in [5]. The proof is similar to that of Lemma 2.1 in [5]. ■

Associated with (1) ($B = 0$, $D_k = 0$), consider its adjoint equation given by

$$\begin{cases} \psi'(t) = -A^* \psi(t), \\ \psi(b) = \varphi, \\ \Delta \psi(t_{p-k+1}) = -C_{p-k+1}^* \psi(t_{p-k+1}^+), \quad k = 1, \dots, p. \end{cases} \quad (2)$$

Here A^* , C_{p-k+1}^* are adjoint operators.

Lemma 2 *The mild solution of the adjoint equation (2) is given by*

$$\begin{aligned} \psi(t) &= S^*(b-t)\varphi, \quad t_p < t \leq b, \\ \psi(t) &= S_C^*(t_k, t) \prod_{i=k+1}^p S_C^*(t_i, t_{i-1}) S^*(b-t_p)\varphi, \quad t_{k-1} < t \leq t_k, \quad k = p, \dots, 1, \quad \prod_{i=p+1}^p = 1. \end{aligned} \quad (3)$$

Proof. For $t_p < t \leq b$ the formula (3) is obvious. For $t_{p-1} < t \leq t_p$, we have

$$\begin{aligned} \psi(t) &= S^*(t_p - t)\psi(t_p^-) = S^*(t_p - t)(I + C_p^*)\psi(t_p^+) \\ &= S^*(t_p - t)(I + C_p^*)S^*(b - t_p)\varphi = S_C^*(t_p, t)S^*(b - t_p)\varphi. \end{aligned}$$

By using the induction we get the desired formula (3). ■

We now introduce the following lemma which characterizes the relationship between solutions of (1), (2) and the control operators.

Lemma 3 For the solutions (1) and (3), the following formula holds:

$$\langle x(b), \psi(b) \rangle - \langle x(0), \psi(0) \rangle = \sum_{k=1}^{p+1} \int_{t_{k-1}}^{t_k} \langle u(s), B^* \psi(s) \rangle ds + \sum_{k=1}^p \langle v_k, D_k^* \psi(t_k^+) \rangle. \quad (4)$$

Proof. It is clear that

$$\langle x(t_1), \psi(t_1) \rangle - \langle x(0), \psi(0) \rangle = \int_0^{t_1} \langle Bu(s), \psi(s) \rangle ds. \quad (5)$$

From Lemmas 1 and 2, we have

$$\begin{aligned} \langle x(b), \psi(b) \rangle - \langle x(t_1), \psi(t_1) \rangle &= \sum_{k=2}^{p+1} [\langle x(t_k), \psi(t_k) \rangle - \langle x(t_{k-1}), \psi(t_{k-1}) \rangle] \\ &= \sum_{k=2}^{p+1} \left[\left\langle S(t_k - t_{k-1}) x(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} S(t_k - s) Bu(s) ds, \psi(t_k) \right\rangle - \langle x(t_{k-1}), (I + C_{k-1}^*) \psi(t_{k-1}^+) \rangle \right] \\ &= \sum_{k=2}^{p+1} \int_{t_{k-1}}^{t_k} \langle Bu(s), S^*(t_k - s) \psi(t_k) \rangle ds \\ &\quad + \sum_{k=2}^{p+1} [\langle x(t_{k-1}) + C_{k-1} x(t_{k-1}) + D_{k-1} v_{k-1}, \psi(t_{k-1}^+) \rangle - \langle x(t_{k-1}), (I + C_{k-1}^*) \psi(t_{k-1}^+) \rangle] \\ &= \sum_{k=2}^{p+1} \int_{t_{k-1}}^{t_k} \langle u(s), B^* \psi(s) \rangle ds + \sum_{k=1}^p \langle v_k, D_k^* \psi(t_k^+) \rangle. \end{aligned} \quad (6)$$

Combining (5) and (6), the formula (4) is obtained. ■

2. Main results

For convenience, denote by $w = (u(\cdot), \{v_k\}_{k=1}^p) \in L^2([0, b], U) \times U^p$. The space $L^2([0, b], U) \times U^p$ of w is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_1$ is defined as $\langle w_1, w_2 \rangle_1 = \int_0^b \langle u_1(s), u_2(s) \rangle_U ds + \sum_{k=1}^p \langle v_{1k}, v_{2k} \rangle_U$ for all $w_1, w_2 \in L^2([0, b], U) \times U^p$.

To define the analogue of controllability operator for impulsive system, we introduce the bounded linear operator $M : L^2([0, b], U) \times U^p \rightarrow H$ as follows

$$\begin{aligned} Mw &= S(b - t_p) \sum_{i=1}^p \prod_{j=p}^{i+1} S_C(t_j, t_{j-1}) \int_{t_{i-1}}^{t_i} S_C(t_i, s) Bu(s) ds + \int_{t_p}^b S(b - s) Bu(s) ds \\ &\quad + S(b - t_p) \sum_{i=2}^p \prod_{j=p}^i S_C(t_j, t_{j-1}) D_{i-1} v_{i-1} + S(b - t_p) D_p v_p. \end{aligned}$$

Letting $x(0) = 0$ in (4) yields $\langle x(b), \varphi \rangle = \langle w, M^* \psi_b \rangle_1 = \int_0^b \langle u(s), B^* \psi(s) \rangle ds + \sum_{k=1}^p \langle v_k, D_k^* \psi(t_k^+) \rangle$, which

implies

$$\begin{aligned} M^* \varphi &= (B^* \psi(\cdot), \{D_k^* \psi(t_k^+)\}_{k=1}^p), \\ B^* \psi(t) &= \begin{cases} B^* S^*(b-t) \varphi, & t_p < t \leq b, \\ B^* S_C^*(t_k, t) \prod_{i=k+1}^p S_C^*(t_i, t_{i-1}) S^*(b-t_p) \varphi, & t_{k-1} < t \leq t_k, \end{cases} \\ D_k^* \psi(t_k^+) &= \begin{cases} D_p^* S^*(b-t_p) \varphi, & k = p, \\ D_k^* \prod_{i=k}^p S_C^*(t_i, t_{i-1}) S^*(b-t_p) \varphi, & k = p-1, \dots, 1. \end{cases} \end{aligned}$$

We are ready to introduce four controllability operators $\Gamma_{t_p}^b, \tilde{\Gamma}_{t_p}^b, \Theta_0^{t_p}, \tilde{\Theta}_0^{t_p} : H \rightarrow H$,

$$\begin{aligned} \Gamma_{t_p}^b &:= \int_{t_p}^b S(b-s) B B^* S^*(b-s) ds, \quad \tilde{\Gamma}_{t_p}^b := S(b-t_p) D_p D_p^* S^*(b-t_p), \\ \Theta_0^{t_p} &:= S(b-t_p) \sum_{i=1}^p \prod_{j=p}^{i+1} S_C(t_j, t_{j-1}) \int_{t_{i-1}}^{t_i} S_C(t_i, s) B B^* S_C^*(t_i, s) ds \prod_{k=i+1}^p S_C^*(t_k, t_{k-1}) S^*(b-t_p), \\ \tilde{\Theta}_0^{t_p} &:= S(b-t_p) \sum_{i=2}^p \prod_{j=p}^i S_C(t_j, t_{j-1}) D_{i-1} D_{i-1}^* \prod_{k=i}^p S_C^*(t_k, t_{k-1}) S^*(b-t_p). \end{aligned}$$

It is clear that $MM^* = \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b$.

Remark 4 Note that in nonimpulsive case $MM^* = \Gamma_0^b := \int_0^b S(b-s) B B^* S^*(b-s) ds$, we have only one controllability operator.

Theorem 5 The following conditions are equivalent.

- (5a) System (1) is approximately controllable on $[0, b]$.
- (5b) $M^* \varphi = 0$ implies that $\varphi = 0$.
- (5c) $\Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b$ is positive.
- (5d) $\varepsilon (\varepsilon I + \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b)^{-1}$ converges to zero operator as $\varepsilon \rightarrow 0^+$ in strong operator topology.
- (5e) $\varepsilon (\varepsilon I + \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b)^{-1}$ converges to zero operator as $\varepsilon \rightarrow 0^+$ in weak operator topology.

Proof. The proof of the equivalence (5a) \iff (5b) is standard. Approximately controllability of system (1) on $[0, b]$ is equivalent to $\text{Im} M$ is dense in H . That means, the kernel of M^* is trivial in H . Equivalently, $M^* \varphi = (B^* \psi(\cdot), \{D_k^* \psi(t_k^+)\}_{k=1}^p) = 0$ implies that $\varphi = 0$. The equivalence (5a) \iff (5c) is well known, see [1] page 207. The equivalence

(5d) \iff (5e) is a consequence of nonnegativity of $\varepsilon \left(\varepsilon I + \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b \right)^{-1}$. We prove only (5a) \iff (5d). To do so, consider the functional

$$J_\varepsilon(\varphi) = \frac{1}{2} \|M^* \varphi\|^2 + \frac{\varepsilon}{2} \|\varphi\|^2 - \left\langle \varphi, h - S(b - t_p) \prod_{j=p}^1 S_C(t_j, t_{j-1}) x_0 \right\rangle$$

The map $\varphi \rightarrow J_\varepsilon(\varphi)$ is continuous and strictly convex. The functional $J_\varepsilon(\cdot)$ admits a unique minimum $\hat{\varphi}_\varepsilon$ that defines a map $\Phi : X \rightarrow X$. Since $J_\varepsilon(\varphi)$ is Frechet differentiable at $\hat{\varphi}_\varepsilon$, by the optimality of $\hat{\varphi}_\varepsilon$, we must have

$$\frac{d}{d\varphi} J_\varepsilon(\varphi) = \Theta_0^{t_p} \hat{\varphi}_\varepsilon + \Gamma_{t_p}^b \hat{\varphi}_\varepsilon + \tilde{\Theta}_0^{t_p} \hat{\varphi}_\varepsilon + \tilde{\Gamma}_{t_p}^b \hat{\varphi}_\varepsilon + \varepsilon \hat{\varphi}_\varepsilon - h + S(b - t_p) \prod_{j=p}^1 S_C(t_j, t_{j-1}) x_0 = 0, \quad (7)$$

By solving (7) for $\hat{\varphi}_\varepsilon$, we get

$$\hat{\varphi}_\varepsilon = \left(\varepsilon I + \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b \right)^{-1} \left(h - S(b - t_p) \prod_{j=p}^1 S_C(t_j, t_{j-1}) x_0 \right). \quad (8)$$

Defining $u^\varepsilon(s)$ and $\{v_k^\varepsilon\}_{k=1}^p$ as follows

$$u^\varepsilon(s) = \left(\sum_{k=1}^p B^* S_C^*(t_k, s) \prod_{i=k+1}^p S_C^*(t_i, t_{i-1}) S^*(b - t_p) \chi_{(t_{k-1}, t_k)} + B^* S^*(b - s) \chi_{(t_p, b)} \right) \hat{\varphi}_\varepsilon,$$

$$v_p^\varepsilon = D_p^* S^*(b - t_p) \hat{\varphi}_\varepsilon, \quad v_k^\varepsilon = D_k^* \prod_{i=k}^p S_C^*(t_i, t_{i-1}) S^*(b - t_p) \hat{\varphi}_\varepsilon, \quad k = 1, \dots, p-1,$$

we get from (7) and (8) that

$$x_\varepsilon(b) - h = -\varepsilon \hat{\varphi}_\varepsilon = -\varepsilon \left(\varepsilon I + \Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b \right)^{-1} \left(h - S(b - t_p) \prod_{j=p}^1 S_C(t_j, t_{j-1}) x_0 \right), \quad (9)$$

where

$$x_\varepsilon(b) = x(b; x_0, u^\varepsilon, \{v_k^\varepsilon\}_{k=1}^p) = S(b - t_p) \prod_{j=p}^1 S_C(t_j, t_{j-1}) x_0 + \Theta_0^{t_p} \hat{\varphi}_\varepsilon + \Gamma_{t_p}^b \hat{\varphi}_\varepsilon + \tilde{\Theta}_0^{t_p} \hat{\varphi}_\varepsilon + \tilde{\Gamma}_{t_p}^b \hat{\varphi}_\varepsilon.$$

Now, the equivalence (5a) \iff (5d) follows immediately from (9). ■

Corollary 6 *If one of the operators $\Theta_0^{t_p}, \Gamma_{t_p}^b, \tilde{\Theta}_0^{t_p}, \tilde{\Gamma}_{t_p}^b$ is positive, then $\Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b$ is positive, and consequently the system (1) is approximately controllable on $[0, b]$.*

Proof. The operators $\Theta_0^{t_p}, \Gamma_{t_p}^b, \tilde{\Theta}_0^{t_p}, \tilde{\Gamma}_{t_p}^b$ are nonnegative. So, if one of them is positive then the sum $\Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b$ is positive. By Theorem 5c, we get the approximate controllability of (1) on $[0, b]$. ■

Corollary 7 Assume $A : H \rightarrow H$ is a linear bounded operator. System (1) is approximately controllable on $[0, b]$ if

$$\overline{\text{sp}\{A^n BU : n = 0, 1, 2, \dots\}} = H. \quad (10)$$

Proof. Suppose by contradiction that

$$\text{Im}M = \{x(b) = x(b; 0, u, \{v_k\}_{k=1}^p) : (u, \{v_k\}_{k=1}^p) \in L^2([0, b], U) \times U^p\}$$

is not dense in H , then for some nonzero $\varphi \in H$ $\langle x(b), \varphi \rangle = 0$:

$$\langle x(b), \varphi \rangle = \int_0^b \langle Bu(s), \psi(s) \rangle ds + \sum_{k=1}^p \langle D_k v_k, \psi(t_k^+) \rangle = 0, \text{ for any } (u, \{v_k\}_{k=1}^p) \in L^2([0, b], U) \times U^p,$$

where ψ is a solution (3) of the adjoint equation with $\varphi \neq 0$. This easily leads to

$$\|M^* \varphi\|^2 = \left\langle \left(\Theta_0^{t_p} + \Gamma_{t_p}^b + \tilde{\Theta}_0^{t_p} + \tilde{\Gamma}_{t_p}^b \right) \varphi, \varphi \right\rangle = 0 \implies \Gamma_{t_p}^b \varphi = 0 \implies B^* S^*(b-s) \varphi = 0, \quad t_p \leq s \leq b.$$

We differentiate successively this last identity to show, by induction, that $B^* \varphi = B^* A^* \varphi = \dots = B^* (A^*)^n \varphi = 0$, $n = 0, 1, 2, \dots$. Therefore $0 \neq \varphi \in \cap_{n=0}^{\infty} \ker \{B^* (A^*)^n\}$. But the condition (10) is equivalent to $\cap_{n=0}^{\infty} \ker \{B^* (A^*)^n\} = 0$ see [13]. This contradiction proves that system (1) is approximately controllable on $[0, b]$. ■

Nonimpulsive analogue of the following wave equation is given in [1].

Theorem 8 If $b - t_p \geq 2\pi$, $\gamma_m \neq 0$ for $m = 1, 2, \dots$, then system

$$\left\{ \begin{array}{l} \frac{\partial^2 x(t, \theta)}{\partial t^2} = \frac{\partial^2 x(t, \theta)}{\partial \theta^2} + hu(t), \\ x(t, 0) = x(t, \pi) = 0, \\ x(0, \theta) = a(\theta), \quad \frac{\partial x(0, \theta)}{\partial t} = b(\theta), \\ \Delta x(t_i, \theta) = a_i(\theta), \quad \Delta \frac{\partial x(t_i, \theta)}{\partial t} = b_i(\theta), \quad i = 1, \dots, p \end{array} \right. \quad (11)$$

is approximately controllable on $[0, b]$.

Proof. We identify functions $a(\theta)$ and $b(\theta)$ with their Fourier expansions

$$a(\theta) = \sum_{m=1}^{\infty} \alpha_m \sin m\theta, \quad b(\theta) = \sum_{m=1}^{\infty} \beta_m \sin m\theta, \quad \theta \in (0, \pi). \quad (12)$$

It is easy to check that

$$x(t, \theta) = \sum_{m=1}^{\infty} \left(\alpha_m \cos mt + \frac{\beta_m}{m} \sin mt \right) \sin m\theta, \quad \frac{\partial x(t, \theta)}{\partial t} = \sum_{m=1}^{\infty} (-m\alpha_m \sin mt + \beta_m \cos mt) \sin m\theta.$$

We define H to be the set of pairs $\begin{bmatrix} a \\ b \end{bmatrix}$ of functions with expansions (12) such that

$$\sum_{m=1}^{\infty} (m^2 |\alpha_m|^2 + |\beta_m|^2) < \infty. \text{ This is a Hilbert space with scalar product } \left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} \right\rangle =$$

$\sum_{m=1}^{\infty} (m^2 \alpha_m \tilde{\alpha}_m + \beta_m \tilde{\beta}_m)$. The semigroup of solutions to the wave equation (11) is defined as follows:

$$S(t) \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} \cos mt & \frac{1}{m} \sin mt \\ -m \sin mt & \cos mt \end{bmatrix} \begin{bmatrix} \alpha_m \\ \beta_m \end{bmatrix} \sin m(\cdot), \quad t \geq 0.$$

The formula is meaningful for all $t \in R$ and $S^*(t) = S^{-1}(t) = S(-t)$, $t \in R$. It is known that the problem can be written as follows:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = S(t) \begin{bmatrix} a \\ b \end{bmatrix} + \int_0^t S(t-s) \begin{bmatrix} 0 \\ h \end{bmatrix} u(s) ds.$$

In our case $U = R$ and the operator $B : R \rightarrow H$ is given by $Bu = \begin{bmatrix} 0 \\ h \end{bmatrix} u$, $u \in R$. Since $S^*(t) = S(-t)$, $t \geq 0$,

$$B^* S^*(b-t) \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{m=1}^{\infty} \gamma_m (m \alpha_m \sin m(b-t) + \beta_m \cos m(b-t)), \quad t_p \leq t \leq b.$$

It is easy to see that the series on the right hand side, when is denoted by $\phi(t)$, $0 \leq t \leq b - t_p$, defines a continuous, periodic function with period 2π . Moreover,

$$m \gamma_m \alpha_m = \frac{1}{\pi} \int_0^{2\pi} \phi(t) \cos mtdt, \quad \gamma_m \beta_m = \frac{1}{\pi} \int_0^{2\pi} \phi(t) \sin mtdt, \quad m = 1, 2, \dots$$

Hence if $b \geq t_p + 2\pi$ and $\phi(t) = 0$ for $0 \leq t \leq b - t_p$, then $m \gamma_m \alpha_m = 0$ and $\gamma_m \beta_m = 0$, $m = 1, 2, \dots$. Since $\gamma_m \neq 0$ for $m = 1, 2, \dots$, $\alpha_m = \beta_m = 0$, $m = 1, 2, \dots$, and we obtain that $a = b = 0$. By Corollary 6, wave equation (11) is approximately controllable. ■

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